

Topic 10 Part 1 [431 marks]

1a. [2 marks]

Markscheme

240, 1020, 3120 **A2**

Note: Award **A2** for three correct answers, **A1** for two correct answers.

[2 marks]

Examiners report

Well answered.

1b. [4 marks]

Markscheme

(i) $1020 = 240 \times 4 + 60$ **(M1)**

$240 = 60 \times 4$

$\gcd(1020, 240) = 60$ **A1**

(ii) $3120 = 1020 \times 3 + 60$ **(M1)**

$1020 = 60 \times 17$

$\gcd(1020, 3120) = 60$ **A1**

Note: Must be done by Euclid's algorithm.

[4 marks]

Examiners report

Also well answered. A few candidates did not use the Euclidean algorithm to find the gcd as instructed.

1c. [1 mark]

Markscheme

by Fermat's little theorem with $p = 5$ **R1**

$n^5 \equiv n \pmod{5}$

so 5 divides $f(n)$

[1 mark]

Examiners report

Many candidates essential said it was true because it was! There is only one mark which means one minute, so it must be a short answer which it is by using Fermat's Little Theorem.

1d. [4 marks]

Markscheme

$$f(n) = n(n^2 - 1)(n^2 + 1) = n(n - 1)(n + 1)(n^2 + 1) \quad (\mathbf{A1})\mathbf{A1}$$

$n - 1$, n , $n + 1$ are consecutive integers and so contain a multiple of 2 and 3 **R1R1**

Note: Award **R1** for justification of 2 and **R1** for justification of 3.

And therefore $f(n)$ is a multiple of 6. **AG**

[4 marks]

Examiners report

Some good answers but too many did not factorize as instructed, so that they could then spot the consecutive numbers.

1e. [3 marks]

Markscheme

from (c) and (d) $f(n)$ is always divisible by 30 **R1**

considering the factorization, it is divisible by 60 when n is an odd number **A1**

or when

n is a multiple of 4 **A1**

Note: Accept answers such as when n is not congruent to 2 (mod 4).

[3 marks]

Total [14 marks]

Examiners report

Only the better candidates realised that they had to find another factor of 2.

2a. [4 marks]

Markscheme

let there be

v vertices in the graph; because the graph is simple the degree of each vertex is $\leq v - 1$ **A1**

the degree of each vertex is ≥ 1 **A1**

there are therefore $v - 1$ possible values for the degree of each vertex **A1**

given there are

v vertices by the pigeon-hole principle there must be at least two with the same degree **R1**

[4 marks]

Examiners report

Generally there were too many “waffly” words and not enough precise statements leading to conclusions.

Misconceptions were: thinking that a few examples constituted a proof, thinking that the graph had to be connected, taking the edges as the pigeons not the degrees. The pigeon-hole principle was known but not always applied well.

2b.

[4 marks]

Markscheme

consider a graph in which the people at the meeting are represented by the vertices and two vertices are connected if the two people shake hands **M1**

the graph is simple as no-one shakes hands with the same person more than once (nor can someone shake hands with themselves) **A1**

every vertex is connected to at least one other vertex as everyone shakes at least one hand **A1**

the degree of each vertex is the number of handshakes so by the proof above there must be at least two who shake the same number of hands **R1**

Note: Accept answers starting afresh rather than quoting part (a).

[4 marks]

Examiners report

Generally there were too many “waffly” words and not enough precise statements leading to conclusions.

Similar problems as in (a).

2c.

[2 marks]

Markscheme

(the handshaking lemma tells us that) the sum of the degrees of the vertices must be an even number **A1**

the degree of each vertex would be

9 and 9×17 is an odd number (giving a contradiction) **A1**

[2 marks]

Total [10 marks]

Examiners report

Generally there were too many “waffly” words and not enough precise statements leading to conclusions.

Many spurious reasons were given but good candidates went straight to the hand-shaking lemma.

3a.

[7 marks]

Markscheme

	A	B	C	D	E	F	G	H	I	J
A	0	10	11	18						
B		10		17	21	23				
C			11							
D				17			24	22		
E					21				30	
H								22	27	
F						23			29	
G							24		28	
J										27

M1A1A1A1

(M1 for an attempt at Dijkstra's)

(A1 for value of $D = 17$)(A1 for value of $H = 22$)(A1 for value of $G = 22$)route is $ABD H J$ (M1)A1

cost is \$27 A1

Note: Accept other layouts.**[7 marks]**

Examiners report

Many candidates had the correct route and the cost. Not all showed sufficient working with their Dijkstra's algorithm. See the mark-scheme for the neat way of laying out the working, including the back-tracking method. This tabular working is efficient, avoids mistakes and saves time.

3b.

[6 marks]

Markscheme

there are 4 odd vertices A, D, F and J A1

these can be joined up in 3 ways with the following extra costs

 AD and FJ $17 + 13 = 30$ AF and DJ $23 + 10 = 33$ AJ and DF $27 + 12 = 39$ M1A1A1**Note:** Award M1 for an attempt to find different routes.

Award A1A1 for correct values for all three costs A1 for one correct.

need to repeat AB, BD, FG and GJ A1total cost is $139 + 30 = \$169$ A1**[6 marks]****Total [13 marks]**

Examiners report

There was often confusion here between this problem and the travelling salesman. Good candidates started with the number of vertices of odd degree. Weaker candidates just tried to write the answer down without complete reasoning. All the 3 ways of joining the odd vertices had to be considered so that you knew you had the smallest. Sometimes a mark was lost by giving which routes (paths) had to be repeated rather than which roads (edges).

4a.

[3 marks]

Markscheme

METHOD 1

listing 9, 20, 31, ... and 1, 6, 11, 16, 21, 26, 31, ... **M1**

one solution is 31 **(A1)**

by the Chinese remainder theorem the full solution is

$$x \equiv 31 \pmod{55} \quad \mathbf{A1 \ N2}$$

METHOD 2

$$x \equiv 9 \pmod{11} \Rightarrow x = 9 + 11t \quad \mathbf{M1}$$

$$\Rightarrow 9 + 11t \equiv 1 \pmod{5}$$

$$\Rightarrow t \equiv 2 \pmod{5} \quad \mathbf{A1}$$

$$\Rightarrow t = 2 + 5s$$

$$\Rightarrow x = 9 + 11(2 + 5s)$$

$$\Rightarrow x = 31 + 55s \ (\Rightarrow x \equiv 31 \pmod{55}) \quad \mathbf{A1}$$

Note: Accept other methods *eg* formula, Diophantine equation.

Note: Accept other equivalent answers e.g. $-79 \pmod{55}$.

[3 marks]

Examiners report

A variety of methods were used here. The Chinese Remainder Theorem method (Method 2 on the mark-scheme) is probably the most instructive. Candidates who tried to do it by formula often (as usual) made mistakes and got it wrong. Marks were lost by just saying 31 and not giving mod (55).

4b.

[4 marks]

Markscheme

$$41^{82} \equiv 8^{82} \pmod{11}$$

by Fermat's little theorem $8^{10} \equiv 1 \pmod{11}$ (or $41^{10} \equiv 1 \pmod{11}$) **M1**

$$8^{82} \equiv 8^2 \pmod{11} \quad \mathbf{M1}$$

$$\equiv 9 \pmod{11} \quad \mathbf{(A1)}$$

remainder is 9 **A1**

Note: Accept simplifications done without Fermat.

[4 marks]

Examiners report

Time was lost here by not using Fermat's Little Theorem as a starting point, although the ad hoc methods will work.

4c. [3 marks]

Markscheme

$$41^{82} \equiv 1^{82} \equiv 1 \pmod{5} \quad \mathbf{A1}$$

so 41^{82} has a remainder 1 when divided by 5 and a remainder 9 when divided by 11 $\mathbf{R1}$

hence by part (a) the remainder is 31 $\mathbf{A1}$

[3 marks]

Total [10 marks]

Examiners report

Although it said use parts (a) and (b) not enough candidates saw the connection.

5a. [1 mark]

Markscheme

$$\frac{1}{2} \quad \mathbf{A1}$$

[1 mark]

Examiners report

Not all candidates wrote this answer down correctly although it was essentially told you in the question.

5b. [4 marks]

Markscheme

Andy could win the n^{th} game by winning the $n - 1^{\text{th}}$ and then winning the n^{th} game or by losing the $n - 1^{\text{th}}$ and then winning the n^{th} $\mathbf{(M1)}$

$$u_n = \frac{1}{2}u_{n-1} + \frac{1}{4}(1 - u_{n-1}) \quad \mathbf{A1A1M1}$$

Note: Award $\mathbf{A1}$ for each term and $\mathbf{M1}$ for addition of two probabilities.

$$u_n = \frac{1}{4}u_{n-1} + \frac{1}{4} \quad \mathbf{AG}$$

[4 marks]

Examiners report

Very badly answered. Candidates seemed to think that they were being told this relationship (so used it to find $u(2)$) rather than attempting to prove it.

5c.

[6 marks]

Markscheme

general solution is $u_n = A\left(\frac{1}{4}\right)^n + p(n)$ **(M1)**

for a particular solution try $p(n) = b$ **(M1)**

$$b = \frac{1}{4}b + \frac{1}{4} \quad \textbf{(A1)}$$

$$b = \frac{1}{3}$$

$$\text{hence } u_n = A\left(\frac{1}{4}\right)^n + \frac{1}{3} \quad \textbf{(A1)}$$

$$\text{using } u_1 = \frac{1}{2} \quad \textbf{M1}$$

$$\frac{1}{2} = A\left(\frac{1}{4}\right) + \frac{1}{3} \Rightarrow A = \frac{2}{3}$$

$$\text{hence } u_n = \frac{2}{3}\left(\frac{1}{4}\right)^n + \frac{1}{3} \quad \textbf{A1}$$

Note: Accept other valid methods.

[6 marks]

Examiners report

This distinguished the better candidates. Some candidates thought that they could use the method for homogeneous recurrence relations of second order and hence started solving a quadratic. Only the better candidates saw that it was a combined AP/GP.

5d.

[2 marks]

Markscheme

for large n $u_n \approx \frac{1}{3}$ **(M1)A1**

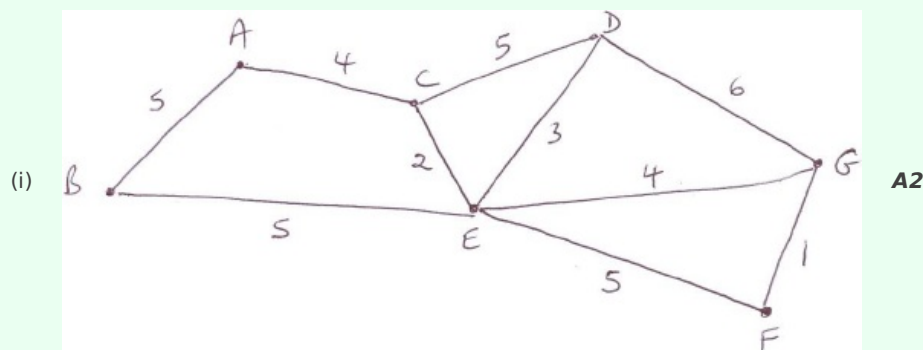
[2 marks]

Total [13 marks]

Examiners report

The best candidates saw this but most had not done enough earlier to get to do this.

Markscheme



Note: Award **A1** if one edge is missing. Award **A1** if the edge weights are not labelled.

(ii) the edges are added in the order:

*FG*1 **M1A1**

*CE*2 **A1**

*ED*3

*EG*4 **A1**

*AC*4

Note:

EG and

AC can be added in either order.

(Reject

EF)

(Reject

CD)

*EB*5 **OR** *AB*5 **A1**

Notes: The minimum spanning tree does not have to be seen.

If only a tree is seen, the order by which edges are added must be clearly indicated.

(iii)

19 **A1**

[8 marks]

Examiners report

Part (a) was generally very well answered. Most candidates were able to correctly sketch the graph of H and apply Kruskal's algorithm to determine the minimum spanning tree of H . A few candidates used Prim's algorithm (which is no longer part of the syllabus).

Markscheme

(i) eg, $PQRSRTSTQP$ OR $PQTSTRSRQP$ **M1A1**

Note: Award **M1** if in either (i) or (ii), it is recognised that edge PQ is needed twice.

(ii) total weight = 34 **A1**

[3 marks]

Examiners report

Most candidates understood the Chinese Postman Problem in part (b) and knew to add the weight of PQ to the total weight of H . Some candidates, however, did not specify a solution to the Chinese Postman Problem while other candidates missed the fact that a return to the initial vertex is required.

Markscheme

(i) to determine a cycle where each vertex is visited once only (Hamiltonian cycle) **A1**
of least total weight **A1**

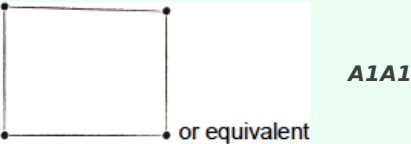
(ii) **EITHER**
to reach P, Q must be visited twice which contradicts the definition of the TSP **R1**
OR
the graph is not a complete graph and hence there is no solution to the TSP **R1**

[3 marks]
Total [14 marks]

Examiners report

In part (c), many candidates had trouble succinctly stating the Travelling Salesman Problem. Many candidates used an ‘edge’ argument rather than simply stating that the Travelling Salesman Problem could not be solved because to reach vertex P , vertex Q had to be visited twice.

Markscheme



Note: Award **A1** for a correct version of $K_{2,2}$ and **A1** for a correct planar representation.
[2 marks]

Examiners report

Part (a) was generally well done with a large number of candidates drawing a correct planar representation for $K_{2,2}$. Some candidates, however, produced a correct non-planar representation of $K_{2,2}$.

7b. [1 mark]

Markscheme



[1 mark]

Examiners report

Parts (b) and (c) were generally well done with many candidates drawing a correct spanning tree for $K_{2,2}$ and the correct complement of $K_{2,2}$.

7c. [1 mark]

Markscheme



[1 mark]

Examiners report

Parts (b) and (c) were generally well done with many candidates drawing a correct spanning tree for $K_{2,2}$ and the correct complement of $K_{2,2}$.

7d. [3 marks]

Markscheme

the complete bipartite graph $K_{m,n}$ has two subsets of vertices A and B , such that each element of A is connected to every element of B **A1**

in the complement, no element of A is connected to any element of B . The complement is not a connected graph **A1**

by definition a tree is connected **R1**

hence the complement of any complete bipartite graph does not possess a spanning tree **AG**

[3 marks]

Total [7 marks]

Examiners report

Part (d) tested a candidate's ability to produce a reasoned argument that clearly explained why the complement of $K_{m,n}$ does not possess a spanning tree. This was a question part in which only the best candidates provided the necessary rigour in explanation.

Markscheme

METHOD 1

attempting to find a solution in the form $u_n = A7^n + B$ **M1**

EITHER

eg, and $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ **A1**

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent) **A1**

THEN

attempting to solve for A and B **(M1)**

$u_n = 4 \times 7^n + 1$ **A1A1**

Note: Accept $A = 4$, $B = 1$ provided the first **M1** is awarded.

METHOD 2

attempting an iterative method eg, $u_1 = 7(5) - 6$ and

$u_2 = 7^2(5) - 6(7 + 1)$ (etc) **(M1)**

$u_n = 5 \times 7^n - 6 \left(\frac{7^n - 1}{7 - 1} \right)$ **M1A1**

Note: Award **M1** for attempting to express u_n in terms of n .

$u_n = 4 \times 7^n + 1$ **A1A1**

METHOD 3

attempting to find a solution in the form $u_n = A7^n + B$ **M1**

$A(n + 1) + B = 7(An + B) - 6$

$7B - 6 = B$ **A1**

attempting to solve for A **(M1)**

$u_n = 4 \times 7^n + 1$ **A1A1**

METHOD 4

$u_{n+1} - 7u_n + 6 - (u_n - 7u_{n+1} + 6) = 0 \Rightarrow u_{n+1} - 8u_n + 7u_{n-1} = 0$

$r^2 - 8r + 7 = 0$

$r = 1, 7$

attempting to find a solution in the form $u_n = A7^n + B$ **M1**

EITHER

eg, $u_0 = 5 \Rightarrow 5 = A + B$ and $u_1 = 29 \Rightarrow 29 = 7A + B$ **A1**

OR

$A7^{n+1} + B = A7^{n+1} + 7B - 6$ (or equivalent) **A1**

THEN

attempting to solve for A and B **(M1)**

$u_n = 4 \times 7^n + 1$ **A1A1**

[5 marks]

Examiners report

In part (a), a good number of candidates were able to 'see' the solution form for u_n and then (often in non-standard ways) successfully obtain $u_n = 4 \times 7^n + 1$. A variety of methods and interesting approaches were seen here including use of the general closed form solution, iteration, substitution of $u_n = 4 \times 7^n + 1$, substitution of $u_n = An + B$ and, interestingly, conversion to a second-degree linear recurrence relation. A number of candidates erroneously converted the recurrence relation to a quadratic auxiliary equation and obtained $u_n = c_1(6)^n + c_2(1)^n$.

8b.

[7 marks]

Markscheme

attempting to find the auxiliary equation **M1**

$$r^2 - 10r - 11 = 0 \quad ((r - 11)(r + 1) = 0) \quad \mathbf{A1}$$

$$r = 11, r = -1 \quad \mathbf{A1}$$

$$v_n = A11^n + B(-1)^n \quad \mathbf{(M1)}$$

attempting to use the initial conditions **M1**

$$A + B = 4, 11A - B = 44 \quad \mathbf{A1}$$

$$v_n = 4 \times 11^n \quad \mathbf{A1}$$

[7 marks]

Examiners report

Compared to similar recurrence relation questions set in recent examination papers, part (b) was reasonably well attempted with a substantial number of candidates correctly obtaining $v_n = 4(11)^n$. It was pleasing to note the number of candidates who could set up the correct auxiliary equation and use the two given terms to obtain the required solution. It appeared that candidates were better prepared for solving second-order linear recurrence relations compared to first-order linear recurrence relations.

8c.

[4 marks]

Markscheme

$$v_n - u_n = 4(11^n - 7^n) - 1 \quad \mathbf{M1}$$

EITHER

$$= 4(11 - 7)(11^{n-1} + \dots + 7^{n-1}) - 1 \quad \mathbf{M1A1}$$

OR

$$= 4((7 + 4)^n - 7^n) - 1 \quad \mathbf{A1}$$

subtracting the 7^n from the expanded first bracket **M1**

THEN

obtaining 16 times a whole number -1 **A1**

$$v_n - u_n \equiv 15 \pmod{16}, n \in \mathbb{N} \quad \mathbf{AG}$$

[4 marks]

Total [16 marks]

Examiners report

Most candidates found part (c) challenging. Only a small number of candidates attempted to either factorise $11^n - 7^n$ or to subtract 7^n from the expansion of $(7 + 4)^n$. It was also surprising how few went for the option of stating that 11 and 7 are congruent mod 4 so $11^n - 7^n \equiv 0 \pmod{4}$ and hence is a multiple of 4.

9a.

[6 marks]

Markscheme

(i) **METHOD 1**attempting to use $f = e - v + 2$ and $e \leq 3v - 6$ (if $v > 2$) **(M1)** $2e \leq 6v - 12 = 6(e - f + 2) - 12$ **M1A1**leading to $2e \geq 3f$ **AG****METHOD 2**each face is bounded by at least three edges **A1****Note:** Award **A1** for stating $e \geq 3f$.each edge either separates two faces or, if an edge is interior to a face, it gets counted twice **R1****Note:** Award **R1** for stating that each edge contributes two to the sum of the degrees of the faces (or equivalent) ie, $\sum \deg(F) = 2e$.adding up the edges around each face **R1**leading to $2e \geq 3f$ **AG**(ii) K_5 has $e = 10$ **A1**if the graph is planar, $f = 7$ **A1**this contradicts the inequality obtained above **R1**hence the graph is non-planar **AG****[6 marks]**

Examiners report

In part (a) (i), many candidates tried to prove $2e \geq 3f$ with numerical examples. Only a few candidates were able to prove this inequality correctly. In part (a) (ii), most candidates knew that K_5 has 10 edges. However, a number of candidates simply drew a diagram with any number of faces and used this particular representation as a basis for their 'proof'. Many candidates did not recognise the 'hence' requirement in part (a) (ii).

9b.

[4 marks]

Markscheme

(i) the sum of the vertex degrees $= 2e$ (or is even) or equivalent **A1**(ii) if each vertex has degree 2, then $2v = 2e$ **A1**substituting $v = e$ into Euler's formula **M1** $f = 2$ **A1****[4 marks]**

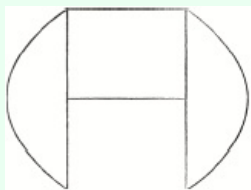
Examiners report

In part (b) (i), many candidates stated the 'handshaking lemma' incorrectly by relating it to the 'handshake problem'. In part (b) (ii), only a few candidates determined that $v = e$ and hence found that $f = 2$.

9c. [2 marks]

Markscheme

for example,



A2

[2 marks]

Total [12 marks]

Examiners report

In part (c), a reasonable number of candidates were able to draw a simple connected planar graph on 6 vertices each of degree 3. The most common error here was to produce a graph that contained a multiple edge(s).

10a. [2 marks]

Markscheme

every positive integer, greater than 1, is either prime or can be expressed uniquely as a product of primes **A1A1**

Note: Award **A1** for “product of primes” and **A1** for “uniquely”.

[2 marks]

Examiners report

In part (a), most candidates omitted the 'uniquely' in their definition of the fundamental theorem of arithmetic. A few candidates defined what a prime number is.

10b. [3 marks]

Markscheme

$$5577 = 3 \times 11 \times 13^2 \text{ and } 99099 = 3^2 \times 7 \times 11^2 \times 13 \quad \mathbf{M1}$$

$$\gcd(5577, 99099) = 3 \times 11 \times 13, \text{ lcm}(5577, 99099) = 3^2 \times 7 \times 11^2 \times 13^2 \quad \mathbf{A1A1}$$

[3 marks]

Examiners report

In part (b), a substantial number of candidates used the Euclidean algorithm rather than the fundamental theorem of arithmetic to calculate $\gcd(5577, 99099)$ and $\text{lcm}(5577, 99099)$.

Markscheme

METHOD 1

$$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r} \text{ and } m = p_1^{j_1} p_2^{j_2} \dots p_r^{j_r} \quad \mathbf{M1}$$

employing all the prime factors of n and m , and inserting zero exponents if necessary $\mathbf{R1}$

define $g_i = \min(k_i, j_i)$ and $h_i = \max(k_i, j_i)$ for $i = 1 \dots r$ $\mathbf{(M1)}$

$$\gcd(n, m) = p_1^{g_1} p_2^{g_2} \dots p_r^{g_r} \text{ and } \text{lcm}(n, m) = p_1^{h_1} p_2^{h_2} \dots p_r^{h_r} \quad \mathbf{A1A1}$$

noting that $g_i + h_i = k_i + j_i$ for $i = 1 \dots r$ $\mathbf{R1}$

$$\gcd(n, m) \times \text{lcm}(n, m) = n \times m \text{ for all } n, m \in \mathbb{Z}^+ \quad \mathbf{AG}$$

METHOD 2

let m and n be expressed as a product of primes where $m = ab$ and $n = ac$ $\mathbf{M1}$

a denotes the factors that are common and b, c are the disjoint factors that are not common $\mathbf{R1}$

$$\gcd(n, m) = a \quad \mathbf{A1}$$

$$\text{lcm}(n, m) = \gcd(n, m)bc \quad \mathbf{A1}$$

$$\gcd(n, m) \times \text{lcm}(n, m) = a \times (abc) \quad \mathbf{M1}$$

$$= ab \times ac \text{ and } m = ab \text{ and } n = ac \text{ so } \mathbf{R1}$$

$$\gcd(n, m) \times \text{lcm}(n, m) = n \times m \text{ for all } n, m \in \mathbb{Z}^+ \quad \mathbf{AG}$$

[6 marks]

Total [11 marks]

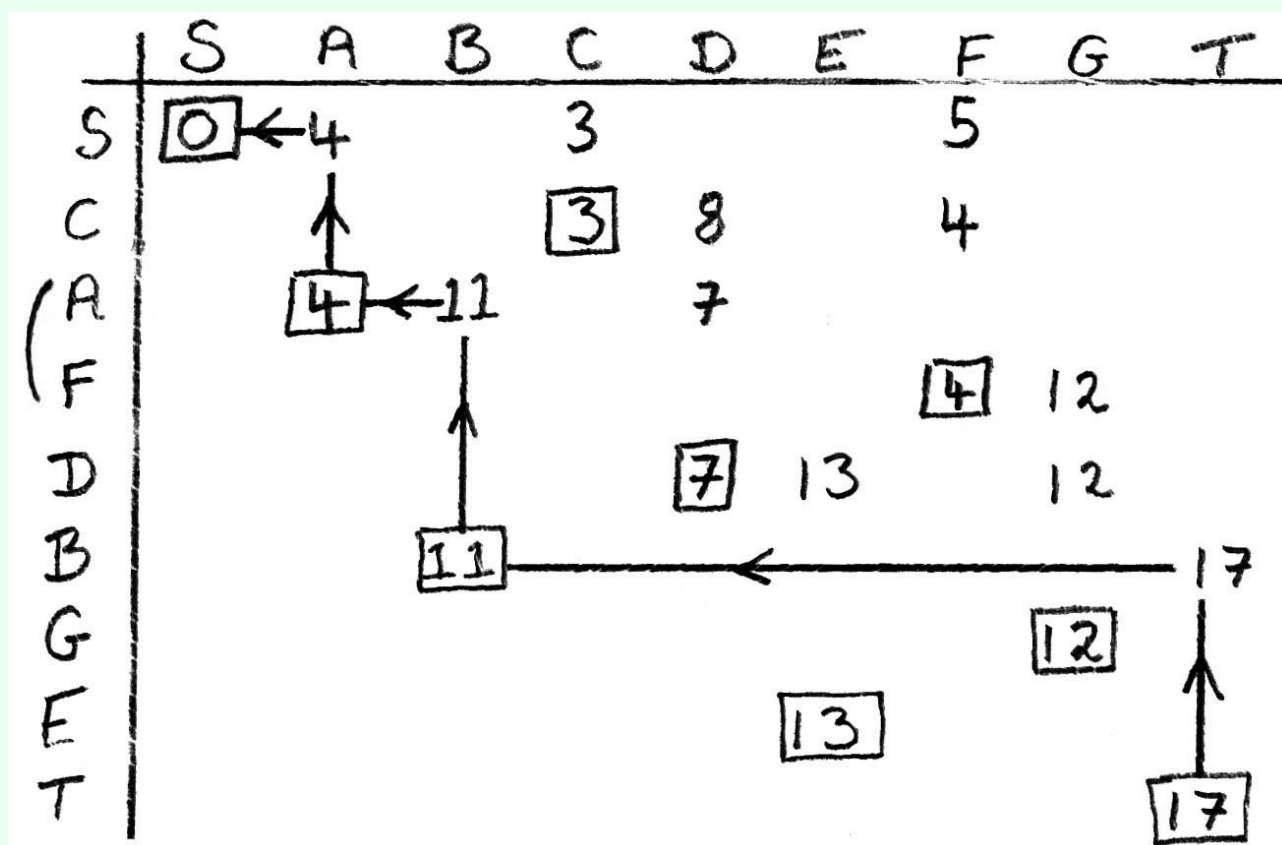
Examiners report

In part (c), a standard proof that has appeared in previous examination papers, was answered successfully by candidates who were well prepared.

11a.

[6 marks]

Markscheme



from tabular method as shown above (or equivalent) *MIAIAI*

Note: Award the first *AI* for obtaining 3 as the shortest distance to C.

Award the second *AI* for obtaining the rest of the shortest distances.

shortest path has length 17 *AI*

backtracking as shown above (or equivalent) *(MI)*

shortest path is SABT *AI*

[6 marks]

Examiners report

This was quite well answered. Some candidates did not make their method clear and others showed no method at all. Some clearly had a correct method but did not make it clear what their final answers were. It is recommended that teachers look at the tabular method with its backtracking system as shown in the mark scheme as an efficient way of tackling this type of problem.

11b.

[4 marks]

Markscheme

(i) no, as *S* and *T* have odd degree *AIRI*

Note: Mentioning one vertex of odd degree is sufficient.

(ii) yes, as only *S* and *T* have odd degree *AIRI*

Note: In each case only award the *AI* if the *RI* has been given.

Accept an actual trail in (b)(ii).

[4 marks]

Examiners report

Fairly good knowledge shown here but not by all.

11c. [4 marks]

Markscheme

Huan has to travel along all the edges via an open Eulerian trail of length **RI**
 $4 + 3 + 5 + 2 + 1 + 3 + 5 + 4 + 7 + 8 + 5 + 6 + 7 + 6 + 6 + 8 + 9 = 94$ **AI**
and then back to S from T along the shortest path found in (a) of length 17 **RI**
so shortest total distance is $94 + 17 = 111$ **AI**
[4 marks]

Examiners report

Some good answers but too much confusion with methods they partly remembered about the travelling salesman problem. Candidates should be aware of helpful connections between parts of a question.

12a. [4 marks]

Markscheme

using the Euclidean Algorithm,

$$332 = 3 \times 99 + 35 \quad \text{MI}$$

$$99 = 2 \times 35 + 29 \quad \text{AI}$$

$$35 = 1 \times 29 + 6$$

$$29 = 4 \times 6 + 5 \quad \text{AI}$$

$$6 = 1 \times 5 + 1 \quad \text{AI}$$

hence 332 and 99 have a gcd of 1 **AG**

Note: For both (a) and (b) accept layout in tabular form, especially the brackets method of keeping track of the linear combinations as the method proceeds.

[4 marks]

Examiners report

Part (a) was well answered and (b) fairly well answered. There were problems with negative signs due to the fact that there was a negative in the question, so candidates had to think a little, rather than just remembering formulae by rote. The lay-out of the algorithm that keeps track of the linear combinations of the first 2 variables is recommended to teachers.

Markscheme

(i) working backwards, (*MI*)

$$6 - 5 = 1$$

$$6 - (29 - 4 \times 6) = 1 \text{ or } 5 \times 6 - 29 = 1 \quad \text{AI}$$

$$5 \times (35 - 29) - 29 = 1 \text{ or } 5 \times 35 - 6 \times 29 = 1 \quad \text{AI}$$

$$5 \times 35 - 6 \times (99 - 2 \times 35) = 1 \text{ or } 17 \times 35 - 6 \times 99 = 1$$

$$17 \times (332 - 3 \times 99) - 6 \times 99 = 1 \text{ or } 17 \times 332 - 57 \times 99 = 1 \quad \text{AI}$$

a solution to the Diophantine equation is therefore

$$x = 17, y = 57 \quad (\text{AI})$$

the general solution is

$$x = 17 + 99N, y = 57 + 332N \quad \text{AIAI}$$

Note: If part (a) is wrong it is inappropriate to give *FT* in (b) as the numbers will contradict, however the *MI* can be given.

(ii) it follows from previous work that

$$17 \times 332 = 1 + 99 \times 57 \quad (\text{MI})$$

$$\equiv 1 \pmod{57} \quad (\text{AI})$$

$$z = 332 \text{ is a solution to the given congruence} \quad (\text{AI})$$

$$\text{the general solution is } 332 + 57N \text{ so the smallest solution is } 47 \quad \text{AI}$$

[11 marks]

Examiners report

Part (a) was well answered and (b) fairly well answered. There were problems with negative signs due to the fact that there was a negative in the question, so candidates had to think a little, rather than just remembering formulae by rote. The lay-out of the algorithm that keeps track of the linear combinations of the first 2 variables is recommended to teachers.

Markscheme

(i) there is an Eulerian trail because there are only 2 vertices of odd degree *RI*

there is no Eulerian circuit because not all vertices have even degree *RI*

(ii) eg GBAGFBCEFCDE *A2*

[4 marks]

Examiners report

In part (a) the criteria for Eulerian circuits and trails were generally well known and most candidates realised that they must start/finish at G/E. Candidates who could not do (a) generally struggled on the paper.

Markscheme

(i)

Step	Vertices labelled	Working values	
1	A	A(0), B-3, G-2	<i>M1A1</i>
2	A, G	A(0), G(2), B-3, F-8	<i>A1</i>
3	A, G, B	A(0), G(2), B(3), F-7, C-10	<i>A1</i>
4	A, G, B, F	A(0), G(2), B(3), F(7), C-9, E-12	
5	A, G, B, F, C	A(0), G(2), B(3), F(7), C(9), E-10, D-15	<i>A1</i>
6	A, G, B, F, C, E	A(0), G(2), B(3), F(7), C(9), E(10), D-14	
7	A, G, B, F, C, E, D	A(0), G(2), B(3), F(7), C(9), E(10), D(14)	<i>A1</i>

Note: In both (i) and (ii) accept the tabular method including back tracking or labels by the vertices on a graph.

Note: Award *M1A1A1A1A0A0* if final labels are correct but intermediate ones are not shown.

(ii) minimum weight path is ABFCED *A1*

minimum weight is 14 *A1*

Note: Award the final two *A1* marks whether or not Dijkstra’s Algorithm is used.

[8 marks]

Examiners report

For part (b) the layout varied greatly from candidate to candidate. Not all candidates made their method clear and some did not show the temporary labels. It is recommended that teachers look at the tabular method with its backtracking system as it is an efficient way of tackling this type of problem and has a very clear layout.

Markscheme

the equation can be written as

$$(3n + 3)^2 = n^3 + 3n^2 + 3n + 1 \quad \textit{M1A1}$$

any valid method of solution giving $n = 8$ *(M1)A1*

Note: Attempt to change at least one side into an equation in n gains the *M1*.

[4 marks]

Examiners report

Part (a) was a good indicator of overall ability. Many candidates did not write both sides of the equation in terms of n and thus had an impossible equation, which should have made them realise that they had a mistake. The answers that were given in (a) and (b) could have been checked, so that the candidate knew they had done it correctly.

14b.

[6 marks]

Markscheme

METHOD 1

as decimal numbers,

$$(33)_8 = 27, (1331)_8 = 729 \quad \text{AIAI}$$

converting to base 7 numbers,

$$27 = (36)_7 \quad \text{AI}$$

$$\begin{array}{r} 7 \overline{)729} \quad \text{MI} \\ 7 \overline{)104} (1 \\ 7 \overline{)14} (6 \\ 7 \overline{)2} (0 \\ 7 \overline{)0} (2 \end{array}$$

therefore

$$729 = (2061)_7 \quad \text{AI}$$

the required equation is

$$36^2 = 2061 \quad \text{AI}$$

METHOD 2

as a decimal number,

$$(33)_8 = 27 \quad \text{AI}$$

converting to base 7,

$$27 = (36)_7 \quad \text{AI}$$

multiplying base 7 numbers

$$\begin{array}{r} 36 \\ \times 36 \\ \hline 1440 \quad \text{MI} \\ \underline{321} \quad \text{AI} \\ 2061 \quad \text{AI} \end{array}$$

the required equation is

$$36^2 = 2061 \quad \text{AI}$$

Note: Allow **MI** for showing the method of converting a number to base 7 regardless of what number they convert.

[6 marks]

Examiners report

Part (b) was not well answered and of those candidates that did, some only gave one side of the equation in base 7. The answers that were given in (a) and (b) could have been checked, so that the candidate knew they had done it correctly.

15a.

[4 marks]

Markscheme

using Fermat’s little theorem,

$$k^p \equiv k(\bmod p) \quad (MI)$$

therefore,

$$\sum_{k=1}^p k^p \equiv \sum_{k=1}^p k(\bmod p) \quad MI$$

$$\equiv \frac{p(p+1)}{2}(\bmod p) \quad AI$$

$$\equiv 0(\bmod p) \quad AG$$

since

$$\frac{(p+1)}{2} \text{ is an integer (so that the right-hand side is a multiple of } p) \quad RI$$

[4 marks]

Examiners report

Only the top candidates were able to produce logically, well thought-out proofs. Too many candidates struggled with the summation notation and were not able to apply Fermat’s little theorem. There was poor logic i.e. looking at a particular example and poor algebra.

15b.

[4 marks]

Markscheme

using the alternative form of Fermat’s little theorem,

$$k^{p-1} \equiv 1(\bmod p), \; 1 \leq k \leq p-1 \quad AI$$

$$k^{p-1} \equiv 0(\bmod p), \; k = p \quad AI$$

therefore,

$$\sum_{k=1}^p k^{p-1} \equiv \sum_{k=1}^{p-1} 1 \; (+0)(\bmod p) \quad MI$$

$$\equiv p-1(\bmod p) \quad AI$$

$$(\text{so } n = p-1)$$

Note: Allow first *AI* even if qualification on *k* is not given.

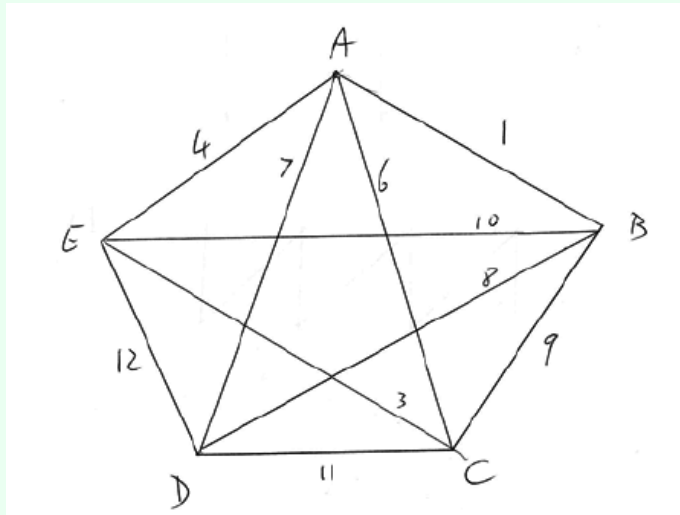
[4 marks]

Examiners report

Only the top candidates were able to produce logically, well thought-out proofs. Too many candidates struggled with the summation notation and were not able to apply Fermat’s little theorem. There was poor logic i.e. looking at a particular example and poor algebra.

16a. [2 marks]

Markscheme



complete graph on five vertices *AI*
weights correctly marked on graph *AI*
[2 marks]

Examiners report

[N/A]

16b. [4 marks]

Markscheme

clear indication that the nearest-neighbour algorithm has been applied *MI*
DA (or 7) *AI*
AB (or 1) BC (or 9) *AI*
CE (or 3), ED (or 12), giving UB = 32 *AI*
[4 marks]

Examiners report

[N/A]

16c. [4 marks]

Markscheme

attempt to use the vertex deletion method *MI*
minimum spanning tree is ECBD *AI*
(EC 3, BD 8, BC 9 total 20)
reconnect A with the two edges of least weight, namely AB (1) and AE (4) *MI*
lower bound is 25 *AI*
[4 marks]

Examiners report

[N/A]

17a. [7 marks]

Markscheme

(i) **METHOD 1**

using a relevant list of powers of 13: (1), 13, 169, (2197) **MI**

$$871 = 5 \times 13^2 + 2 \times 13 \quad \textbf{AI}$$

$$871 = 520_{13} \quad \textbf{AI}$$

$$1157 = 6 \times 13^2 + 11 \times 13 \quad \textbf{AI}$$

$$1157 = 6B0_{13} \quad \textbf{AI}$$

METHOD 2

attempted repeated division by 13 **MI**

$$871 \div 13 = 67; 67 \div 13 = 5 \text{rem} 2 \quad \textbf{AI}$$

$$871 = 520_{13} \quad \textbf{AI}$$

$$1157 \div 13 = 89; 89 \div 13 = 6 \text{rem} 11 \quad \textbf{AI}$$

$$1157 = 6B0_{13} \quad \textbf{AI}$$

Note: Allow (11) for B only if brackets or equivalent are present.

(ii)

$$871 = 13 \times 67; 1157 = 13 \times 89 \quad \textbf{(MI)}$$

67 and 89 are primes (base 10) or they are co-prime **AI**

So

$$\gcd(871, 1157) = 13 \quad \textbf{AG}$$

Note: Must be done by hence not Euclid's algorithm on 871 and 1157.

[7 marks]

Examiners report

[N/A]

17b. [4 marks]

Markscheme

let K be the set of possible remainders on division by n **(MI)**

then

$K = \{0, 1, 2, \dots, n-1\}$ has n members **AI**

because

$$n < n+1 \quad (=n(L)) \quad \textbf{AI}$$

by the pigeon-hole principle (appearing anywhere and not necessarily mentioned by name as long as is explained) **RI**

at least two members of L correspond to one member of K **AG**

[4 marks]

Examiners report

[N/A]

Markscheme

(i) form the appropriate linear combination of the equations: *(MI)*

$$2a + b - c = 7x + 7z \quad AI$$

$$= 7(x + z) \quad RI$$

so 7 divides

$$2a + b - c \quad AG$$

(ii) modulo 2, the equations become *MI*

$$y + z = 1$$

$$z = 0 \quad AI$$

$$x = 1$$

$$\text{solution: } (1, 1, 0) \quad AI$$

Note: Award full mark to use of GDC (or done manually) to solve the system giving $x = -1, y = -3, z = 2$ and then converting mod 2.

[6 marks]

Examiners report

[N/A]

Markscheme

(i) separate consideration of even and odd n *MI*

$$\text{even}^2 - \text{even} + \text{odd is odd} \quad AI$$

$$\text{odd}^2 - \text{odd} + \text{odd is odd} \quad AI$$

$$\text{all elements of } P \text{ are odd} \quad AG$$

Note: Allow other methods *eg*, $n^2 - n = n(n - 1)$ which must be even.

(ii) the list is [41, 41, 43, 47, 53, 61] *AI*

(iii)

$$41^2 - 41 + 41 = 41^2 \text{ divisible by } 41 \quad AI$$

but is not a prime *RI*

the statement is disproved (by counterexample) *AG*

[6 marks]

Examiners report

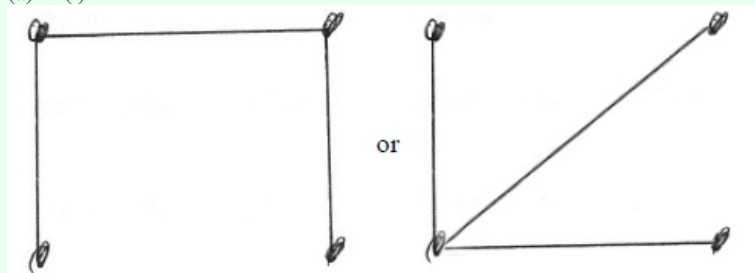
[N/A]

18.

[10 marks]

Markscheme

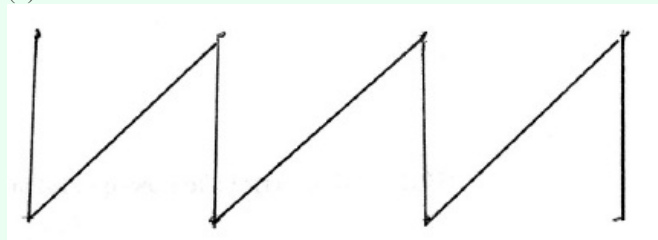
(a) (i)



AI

Note: Or equivalent not worrying about the orientation.

(ii)



AI

Note: Other trees are possible, but must clearly come from the bipartite graph, so, for example, a straight line graph is not acceptable unless the bipartite nature is clearly shown *eg*, with black and white vertices.

[2 marks]

(b) graph is simple implies maximum degree is

 $n - 1$ AI

graph is connected implies minimum degree is 1 AI

by a pigeon-hole principle two vertices must have the same degree RI

[3 marks]

(c) if the graph is not a tree it contains a cycle AI

remove one edge of this cycle MI

the graph remains connected AI

repeat until there are no cycles MI

the final graph is connected and has no cycles AI

so is a tree AG

Note: Allow other methods *eg*, induction, reference to Kruskal's algorithm.

[5 marks]

Total [10 marks]

Examiners report

[N/A]

Markscheme

(a) (i) use of auxiliary equation or recognition of a geometric sequence **(M1)**

$$u_n = (-2)^n u_0 \text{ or}$$

$$= A(-2)^n \text{ or}$$

$$u_1(-2)^{n-1} \quad \textbf{A1}$$

(ii) substitute suggested solution **MI**

$$An + B + 2(A(n-1) + B) = 3n - 2 \quad \textbf{A1}$$

equate coefficients of powers of n and attempt to solve **(M1)**

$$A = 1, B = 0 \quad \textbf{A1}$$

(so particular solution is

$$u_n = n)$$

(iii) use of general solution = particular solution + homogeneous solution **(M1)**

$$u_n = C(-2)^2 + n \quad \textbf{A1}$$

attempt to find C using

$$u_1 = 7 \quad \textbf{M1}$$

$$u_n = -3(-2)^n + n \quad \textbf{A1}$$

[10 marks]

(b) the auxiliary equation is

$$r^2 - 2r + 2 = 0 \quad \textbf{A1}$$

solutions:

$$r_1, r_2 = 1 \pm i \quad \textbf{A1}$$

general solution of the recurrence:

$$u_n = A(1+i)^n + B(1-i)^n \text{ or trig form} \quad \textbf{A1}$$

attempt to impose initial conditions **MI**

$$A = B = 1 \text{ or corresponding constants for trig form} \quad \textbf{A1}$$

$$u_n = 2^{\left(\frac{n}{2}+1\right)} \times \cos\left(\frac{n\pi}{4}\right) \quad \textbf{A1A1}$$

[7 marks]

Total [17 marks]

Examiners report

[N/A]

Markscheme

(a) use Kruskal's algorithm: begin by choosing the shortest edge and then select a sequence of edges of non-decreasing weights, checking at each stage that no cycle is completed (M1)

choice edge weight

- | | | |
|---|----|---|
| 1 | BG | 1 |
| 2 | AG | 2 |
| 3 | FG | 3 |
| 4 | BC | 4 |
| 5 | DE | 5 |
| 6 | AH | 6 |
| 7 | EG | 7 |

A1

A3

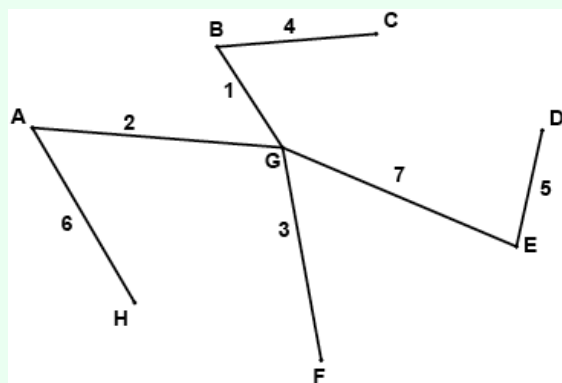
Note: A1 for steps 2–4, A1 for step 5 and A1 for steps 6, 7.

Award marks only if it is clear that Kruskal's algorithm is being used.

[5 marks]

(b) weight

$$= 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \quad \text{A1}$$



A1

Note: Award FT only if it is a spanning tree.

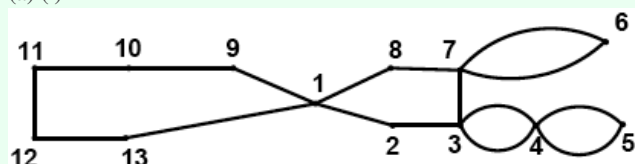
[2 marks]

Examiners report

Well answered by the majority of candidates.

Markscheme

(a) (i)



(MI)AI

Note: Do not penalize candidates who include the entrance foyer.

(ii) the degrees of the vertices are 4, 2, 4, 4, 2, 2, 4, 2, 2, 2, 2, 2, 2 AI

(iii) the degree of all vertices is even and hence a Eulerian circuit exists, AI

hence it is possible to enter the museum through the foyer and visit each room 1–13 going through each internal doorway exactly once AG

Note: The connected graph condition is not required.

[4 marks]

(b) (i)

G							total	H							total
	A	B	C	D	E	F			P	Q	R	S	T	U	
A	0	2	0	2	0	0	4	P	0	1	3	0	1	2	7
B	2	0	1	1	0	1	5	Q	1	0	1	3	2	0	7
C	0	1	0	1	2	1	5	R	3	1	0	2	1	3	10
D	2	1	1	0	2	0	6	S	0	3	2	0	2	0	7
E	0	0	2	2	0	2	6	T	1	2	1	2	0	1	7
F	0	1	1	0	2	0	4	U	2	0	3	0	1	0	6
total							30	total							44

(MI)

graph G has 15 edges and graph H has 22 edges AIAI

(ii) the degree of every vertex is equal to the sum of the numbers in the corresponding row (or column) of the adjacency table exactly two of the vertices of G have an odd degree (B and C) AI

H has four vertices with odd degree AI

G is the graph that has a Eulerian trail (and H does not) RI

(iii) neither graph has all vertices of even degree RI

therefore neither of them has a Eulerian circuit AG

[7 marks]

Examiners report

Part (a) was generally well answered. There were many examples of full marks in this part. Part (b) caused a few more difficulties, although there were many good solutions. Few candidates used the matrix to find the number of edges, preferring instead to draw the graph. A surprising number of students confused the ideas of having vertices of odd degree.

Markscheme

(a)

$$10 \equiv 1 \pmod{9} \Rightarrow 10^i \equiv 1 \pmod{9}, i = 1, \dots, n \quad \text{MIAI}$$

$$\Rightarrow 10^i a_i \equiv a_i \pmod{9}, i = 1, n \quad \text{MI}$$

Note: Allow $i = 0$ but do not penalize its omission.

$$\Rightarrow (10^n a_n + 10^{n-1} a_{n-1} + \dots + a_0) \equiv (a_n + a_{n-1} + \dots + a_0) \pmod{9} \quad \text{AG}$$

[3 marks]

(b)

$$4 + 7 + 6 + x + 2 + 1 + 2 + y = 9k, k \in \mathbb{Z} \quad \text{(MI)}$$

$$\Rightarrow (22 + x + y) \equiv 0 \pmod{9}, \Rightarrow (x + y) \equiv 5 \pmod{9}$$

$$\Rightarrow x + y = 5 \text{ or}$$

$$14 \quad \text{AI}$$

if

5 divides

 a , then

$$y = 0 \text{ or}$$

$$5 \quad \text{MI}$$

so

$$y = 0 \Rightarrow x = 5, (ie (x, y) = (5, 0)) \quad \text{AI}$$

$$y = 5 \Rightarrow x = 0 \text{ or}$$

$$x = 9, (ie (x, y) = (0, 5) \text{ or } (x, y) = (9, 5)) \quad \text{AIAI}$$

[6 marks]

(c) (i)

34390	1
3821	5
424	1
47	2
5	

(MI)AI

$$b = (52151)_9 \quad \text{AG}$$

(ii)

					5	2	1	5	1	
					×	5	2	1	5	1
					<hr/>					
					5	2	1	5	1	
2				8	1	7	7	5		
5				2	1	5	1			
1			1	4	3	1	2			
2	8	1	7	7	5					
<hr/>										
3	0	4	2	3	5	8	1	1	1	

MIA3

Note: **MI** for attempt, **AI** for two correct lines of multiplication, **A2** for two correct lines of multiplication and a correct addition, **A3** for all correct.

[6 marks]

Examiners report

Surprisingly few good answers. Part (a) had a number of correct solutions, but there were also many that seemed to be a memorised solution, not properly expressed – and consequently wrong. In part (b) many failed to understand the question, not registering that x and y were digits rather than numbers. Part (c)(i) was generally well answered, although there were a number of longer methods applied, and few managed to do (c)(ii).

23a. [3 marks]

Markscheme

eg the cycle

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ is Hamiltonian **AI**

starting from any vertex there are four choices

from the next vertex there are three choices, *etc* ... **RI**

so the number of Hamiltonian cycles is

$4! (= 24)$ **AI**

Note: Allow 12 distinct cycles (direction not considered) or 60 (if different starting points count as distinct). In any case, just award the second **AI** if **RI** is awarded.

[3 marks]

Examiners report

Part (a) was generally well answered, with a variety of interpretations accepted.

23b. [1 mark]

Markscheme

total weight of any Hamiltonian cycles stated

eg

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ has weight

$5 + 6 + 7 + 8 + 9 = 35$ **AI**

[1 mark]

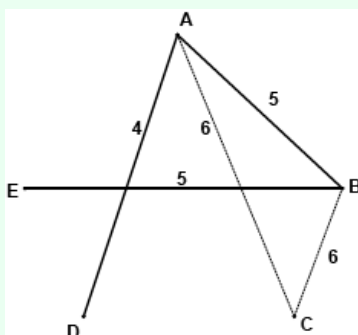
Examiners report

Part (b) also had a number of acceptable possibilities.

23c. [2 marks]

Markscheme

a lower bound for the travelling salesman problem is then obtained by adding the weights of CA and CB to the weight of the minimum spanning tree **(M1)**



a lower bound is then

$14 + 6 + 6 = 26$ **AI**

[2 marks]

Examiners report

Part (d) was generally well answered, but there were few good attempts at part (e).

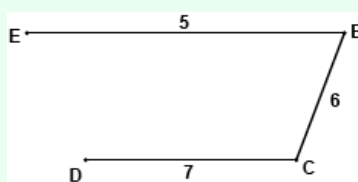
23d.

[3 marks]

Markscheme

METHOD 1

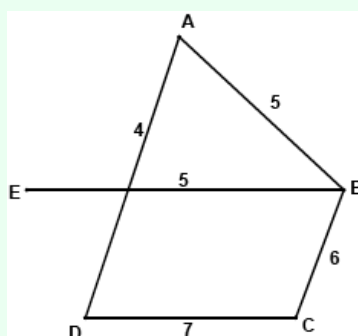
eg eliminating A from G, a minimum spanning tree of weight 18 is *(MI)*



AI

adding AD and AB to the spanning tree gives a lower bound of

$$18 + 4 + 5 = 27 > 26 \quad \text{AI}$$



so 26 is not the best lower bound *AG*

Note: Candidates may delete other vertices and the lower bounds obtained are B-28, D-27 and E-28.

METHOD 2

there are 12 distinct cycles (ignoring direction) with the following lengths

Cycle Length

ABCDEA 35 *MI*

ABCEDA 33

ABDCEA 39

ABDECA 37

ABECDA 31

ABEDCA 31

ACBDEA 37

ACBEDA 29

ACDBEA 35

ACEBDA 33

AEBCDA 31

AECBDA 37 *AI*

as the optimal solution has length 29 *AI*

26 is not the best possible lower bound *AG*

Note: Allow answers where candidates list the 24 cycles obtained by allowing both directions.

[3 marks]

Examiners report

Part (d) was generally well answered, but there were few good attempts at part (e).

Markscheme

METHOD 1

$$n^5 - n = \underbrace{n(n-1)(n+1)}_{\text{3 consecutive integers}} (n^2 + 1) \equiv 0 \pmod{6} \quad \textbf{MI}$$

at least a factor is multiple of 3 and at least a factor is multiple of 2 **RI**

$$n^5 - n = n(n^4 - 1) \equiv 0 \pmod{5} \text{ as}$$

$$n^4 \equiv 1 \pmod{5} \text{ by FLT} \quad \textbf{RI}$$

therefore, as

$$(5, 6) = 1, \quad \textbf{RI}$$

$$n^5 - n \equiv 0 \pmod{\underbrace{5 \times 6}_{30}} \quad \textbf{AI}$$

ie 30 is a factor of

$$n^5 - n \quad \textbf{AG}$$

METHOD 2

let

$P(n)$ be the proposition:

$$n^5 - n = 30\alpha \text{ for some}$$

$$\alpha \in \mathbb{Z}$$

$$0^5 - 0 = 30 \times 0, \text{ so}$$

$$P(0) \text{ is true} \quad \textbf{AI}$$

assume

$P(k)$ is true for some

k and consider

$$P(k+1)$$

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \quad \textbf{MI}$$

$$= (k^5 - k) + 5k \left(\underbrace{k^3 + 3k^2 + 3k + 1}_{(k+1)^3} - (k^2 + k) \right)$$

$$= (k^5 - k) + 5k \left((k+1)^3 - k(k+1) \right)$$

$$= 30\alpha + 5k(k+1) \underbrace{\left(\underbrace{k^2 + k + 1}_{k(k+1)+1} \right)}_{\text{multiple of 6}} \quad \textbf{MI}$$

$$= 30\alpha + 30\beta \quad \textbf{AI}$$

as

$P(0)$ is true and

$P(k)$ true implies

$P(k+1)$ true, by PMI

$P(n)$ is true for all values

$$n \in \mathbb{N} \quad \textbf{RI}$$

Note: Award the first **MI** only if the correct induction procedure is followed and the correct first line is seen.

Note: Award **RI** only if both **M** marks have been awarded.

METHOD 3

$$n^5 - n = n(n^4 - 1) \quad \textbf{MI}$$

$$= n(n^2 - 1)(n^2 + 1) \quad \textbf{AI}$$

$$= (n-1)n(n+1)(n^2 - 4 + 5) \quad \textbf{RI}$$

$$= (n-2)(n-1)n(n+1)(n+2) + 5(n-1)n(n+1) \quad \textbf{AI}$$

each term is multiple of 2, 3 and 5 **RI**

therefore is divisible by 30 **AG**

[5 marks]

Examiners report

Many students were able to get partial credit for part (a), although few managed to gain full marks.

Markscheme

(i) **METHOD 1**

case 1:

$m = 0$ and

$3^0 \equiv 3 \pmod{4}$ is true **AI**

case 2:

$m > 0$

let

$N = 3^m \geq 3$ and consider the binomial expansion **MI**

$$3^N = (1+2)^N = \sum_{k=0}^N \binom{N}{k} 2^k \equiv 1 + 2N + \underbrace{\sum_{k=2}^N \binom{N}{k} 2^k}_{\equiv 0 \pmod{4}} \equiv 1 + 2N \pmod{4} \quad \text{AI}$$

as

$$\underbrace{3^m}_N \equiv (-1)^m \pmod{4} \Rightarrow 1 + 2N \equiv 1 + 2(-1)^m \pmod{4} \quad \text{RI}$$

therefore

$$\underbrace{3^{3^m}}_{3^N} \equiv 1 + 2(-1)^m \pmod{4} \Rightarrow \begin{cases} \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1+2}_3 \pmod{4} \text{ for } m \text{ even} \\ \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1-2}_{-1 \equiv 3 \pmod{4}} \pmod{4} \text{ for } m \text{ odd} \end{cases} \quad \text{RI}$$

which proves that

$3^{3^m} \equiv 3 \pmod{4}$ for any

$m \in \mathbb{N}$ **AG**

METHOD 2

let

$P(n)$ be the proposition:

$3^{3^n} - 3 \equiv 0 \pmod{4, \text{ or } 24}$

$3^{3^0} - 3 = 3 - 3 \equiv 0 \pmod{4 \text{ or } 24}$, so

$P(0)$ is true **AI**

assume

$P(k)$ is true for some

k **MI**

consider

$$3^{3^{k+1}} - 3 = 3^{3^k \times 3} - 3 \quad \text{MI}$$

$$= (3 + 24r)^3 - 3$$

$$\equiv 27 + 24t - 3 \quad \text{RI}$$

$$\equiv 0 \pmod{4 \text{ or } 24}$$

as

$P(0)$ is true and

$P(k)$ true implies

$P(k+1)$ true, by PMI

$P(n)$ is true for all values

$n \in \mathbb{N}$ **RI**

METHOD 3

$$3^{3^m} - 3 = 3(3^{3^m-1} - 1) \quad \text{MIAI}$$

$$= 3(3^{2k} - 1) \quad \text{RI}$$

$$= 3(9^k - 1)$$

$$= 3 \underbrace{\left((8+1)^k - 1 \right)}_{\text{multiple of 8}} \quad \text{RI}$$

$$\equiv 0 \pmod{24} \quad \text{AI}$$

which proves that

$$3^{3^m} \equiv 3 \pmod{4} \text{ for any}$$

$$m \in \mathbb{N} \quad \textbf{AG}$$

(ii) for

$$m \in \mathbb{N}, 3^{3^m} \equiv 3 \pmod{4} \text{ and, as}$$

$$2^{2^n} \equiv 0 \pmod{4} \text{ and}$$

$$5^2 \equiv 1 \pmod{4} \text{ then}$$

$$2^{2^n} + 5^2 \equiv 1 \pmod{4} \text{ for}$$

$$n \in \mathbb{Z}^+$$

there is no solution to

$$3^{3^m} = 2^{2^n} + 5^2 \text{ for pairs}$$

$$(m, n) \in \mathbb{N} \times \mathbb{Z}^+ \quad \textbf{RI}$$

when

$$n = 0, \text{ we have}$$

$$3^{3^m} = 2^{2^0} + 5^2 \Rightarrow 3^{3^m} = 27 \Rightarrow m = 1 \quad \textbf{MI}$$

therefore

$$(m, n) = (1, 0) \quad \textbf{AI}$$

is the only pair of non-negative integers that satisfies the equation \textbf{AG}

[8 marks]

Examiners report

There seemed to be very few good attempts at part (b), many failing at the outset to understand what was meant by 3^{3^m} .

25a.

[8 marks]

Markscheme

$$2347 = 19 \times 123 + 10 \quad \textbf{MIAI}$$

$$(123 = 12 \times 10 + 3)$$

$$10 = 3 \times 3 + 1 \quad \textbf{AI}$$

$$1(\text{gcd}) = 10 - 3 \times 3 = 10 - 3 \times (123 - 12 \times 10) \quad \textbf{MIAI}$$

$$= 37 \times 10 - 3 \times 123 \quad \textbf{AI}$$

$$= 37 \times (2347 - 19 \times 123) - 3 \times 123 \text{ (for continuation)} \quad \textbf{MI}$$

$$= 37 \times 2347 - 706 \times 123 \quad \textbf{AI}$$

[8 marks]

Examiners report

The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

25b.

[3 marks]

Markscheme

EITHER

$$1 \pmod{2347} = (-706 \times 123) \pmod{2347} \quad \textbf{MIAI}$$

OR

$$x = -706 + 2347n \quad \textbf{MIAI}$$

$$\text{solution: } 1641 \quad \textbf{AI}$$

[3 marks]

25c. [3 marks]

Markscheme

$$5(\text{ mod } 2347) = (-3530 \times 123)(\text{ mod } 2347) \quad (M1)$$

$$\text{GS: } z = -3530 + k2347 \quad A1A1$$

Note: Other common possibilities include

$$1164 + k2347 \text{ and}$$

$$8205 + k2347.$$

[3 marks]

Examiners report

The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

25d. [1 mark]

Markscheme

empty set (123 and 2346 both divisible by 3) $A1$

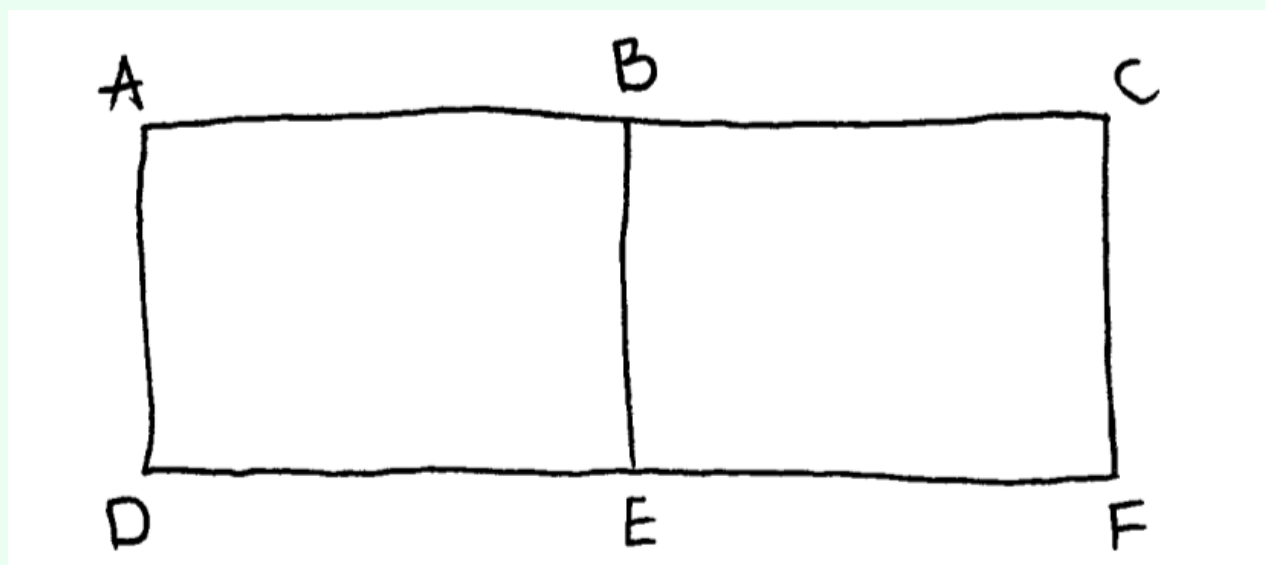
[1 mark]

Examiners report

The majority of candidates were successful in parts (a) and (b). In part (c), some candidates failed to understand the distinction between a particular solution and a general solution. Part (d) was a 1 mark question that defeated all but the few who noticed that the gcd of the numbers concerned was 3.

26a. [2 marks]

Markscheme



A2

Note: Award $A1$ if only one error, $A0$ for two or more.

[2 marks]

Examiners report

Parts (a) and (c) were generally correctly answered. In part (b), a minority of candidates failed to mention that the starting and end points had to coincide. A large number of candidates gave all walks (trails were asked for) – an unnecessary loss of marks.

26b. [1 mark]

Markscheme

the (k, k) element of M

2 is the number of vertices directly connected to vertex k AI

Note: Accept comment about the number of walks of length 2, in which the initial and final vertices coincide.

[1 mark]

Examiners report

Parts (a) and (c) were generally correctly answered. In part (b), a minority of candidates failed to mention that the starting and end points had to coincide. A large number of candidates gave all walks (trails were asked for) – an unnecessary loss of marks.

26c. [3 marks]

Markscheme

the trails of length 4 are ABEDC, AFEDC, AFEBC $AIAIAI$

Note: $AIAIAI$ for three correct with no additions; $AIAIAI0$ for all correct, but with additions; $AIA0A0$ for two correct with or without additions.

[3 marks]

Examiners report

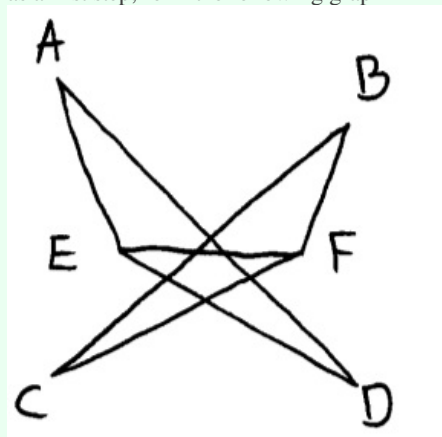
Parts (a) and (c) were generally correctly answered. In part (b), a minority of candidates failed to mention that the starting and end points had to coincide. A large number of candidates gave all walks (trails were asked for) – an unnecessary loss of marks.

27a.

[3 marks]

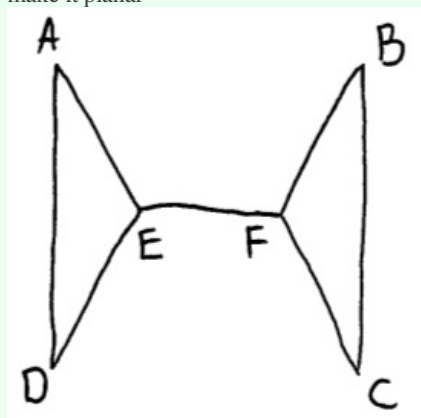
Markscheme

as a first step, form the following graph



$(MI)(AI)$

make it planar



AI

[3 marks]

Examiners report

Part (a) was well done. The various parts of Parts (b) were often attempted, but with a disappointing feeling that the candidates did not have a confident understanding of what they were writing.

27b. [14 marks]

Markscheme

(i) an edge joins a pair of vertices **RI**
 there is a maximum of
 $\binom{v}{2} = \frac{1}{2}v(v-1)$ possible unordered pairs of vertices, hence displayed result **AIAG**

(ii) an edge joins two vertices in G' if it does not join them in G and *vice versa*; all possible edges are accounted for by the union of the two graphs **RI**
 $e + e' = \frac{1}{2}v(v-1)$ **AI**

(iii) the two graphs have the same number of edges **RI**
 $\Rightarrow e = \frac{1}{4}v(v-1)$ **AI**
 v and $v-1$ are consecutive integers, so only one can be divisible by 4, hence displayed result **AIAG**

(iv) the required graphs have four vertices and three edges **AI**
 if one vertex is adjacent to the other three, that uses up the edges; the resulting graph, necessarily connected, has a disconnected complement, and *vice versa* **RI**
 if one vertex is adjacent to two others, that uses up two edges; the final vertex cannot be adjacent to the first; the result is the linear connected graph **AI**
 state it is isomorphic to its complement **AI N2**
Note: Alternative proofs are possible, but should include the final statement for full marks.

(v) using
 $e \leq 3v - 6$ for planar graphs **MI**
 $\frac{1}{2}v(v-1) = e + e' \leq 6v - 12$ **AI**
 $v^2 - 13v + 24 \leq 0$ is not possible for
 $v \geq 11$ **RI**
[14 marks]

Examiners report

Part (a) was well done. The various parts of Parts (b) were often attempted, but with a disappointing feeling that the candidates did not have a confident understanding of what they were writing.

28a. [3 marks]

Markscheme

$2^{2003} = 2^5 \times (2^6)^{333}$ **MIAI**
 $\equiv 32 \times 1 \pmod{7}$ by Fermat's little theorem **AI**
 $\equiv 4 \pmod{7}$ **AG**
[3 marks]

Examiners report

Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

28b. [3 marks]

Markscheme

$2003 = 3 + 10 \times 200$ **(MI)**
 $2^{2003} = 2^3 \times (2^{10})^{200} (\equiv 8 \times 1 \pmod{11}) \equiv 8 \pmod{11}$ **AI**
 $2^{2003} = 2^{11} \times (2^{12})^{166} \equiv 7 \pmod{13}$ **AI**
[3 marks]

Examiners report

Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

28c. [7 marks]

Markscheme

form

$$M_1 = \frac{1001}{7} = 143; M_2 = \frac{1001}{11} = 91; M_3 = \frac{1001}{13} = 77 \quad \text{MI}$$

solve

$$143x_1 \equiv 1 \pmod{7} \Rightarrow x_1 = 5 \quad \text{MIAI}$$

$$x_2 = 4; x_3 = 12 \quad \text{AIAI}$$

$$x = 4 \times 143 \times 5 + 8 \times 91 \times 4 + 7 \times 77 \times 12 = 12240 \equiv 228 \pmod{1001} \quad \text{MIAI}$$

[7 marks]

Examiners report

Many candidates were able to complete part (a) and then went on to part (b). Some candidates raced through part (c). Others, who attempted part (c) using the alternative strategy of repeatedly solving linear congruencies, were sometimes successful.

29a. [4 marks]

Markscheme

METHOD 1

861	957	[1, 0]	[0, 1]
	-1×861		
	<u>-861</u>		<u>-[1, 0]</u>
	96		[-1, 1]
-8×96	<u>-768</u>		
	93		<u>-8[-1, 1]</u>
			[9, -8]
	-1×93		
	<u>-93</u>		<u>-[9, -8]</u>
	3		[-10, 9]

by the above working on the left (or similar) *MIAIAI*

Note: Award *AI* for 96 and *AI* for 93.

$$h = 3 \text{ (since 3 divides 93)} \quad \text{AI}$$

[4 marks]

METHOD 2

$$957 = 861 + 96 \quad \text{MIAI}$$

$$861 = 8 \times 96 + 93 \quad \text{AI}$$

$$96 = 93 + 3$$

$$\text{so } h = 3 \text{ (since 3 divides 93)} \quad \text{AI}$$

[4 marks]

Examiners report

The Euclidean algorithm was well applied. If it is done in the format shown in the mark scheme then the keeping track method of the linear combinations of the 2 original numbers makes part (b) easier.

29b. [3 marks]

Markscheme

METHOD 1

if method 1 was used for part (a)

by the above working on the right (or equivalent) *MI*

$$-10 \times 861 + 9 \times 957 = 3$$

so $A = -10$ and $B = 9$ *AIAI*

[3 marks]

METHOD 2

$$3 = 96 - 93$$

$$= 96 - (861 - 8 \times 96) = 9 \times 96 - 861 \quad \text{MI}$$

$$= 9 \times (957 - 861) - 861$$

$$= -10 \times 861 + 9 \times 957$$

so $A = -10$ and $B = 9$ *AIAI*

[3 marks]

Examiners report

Again well answered but not quite as good as (a).

29c. [5 marks]

Markscheme

from (b)

$$-10 \times 287 + 9 \times 319 = 1 \text{ so } \quad \text{AI}$$

$$-10 \times 287 \equiv 1 \pmod{319} \quad \text{AI}$$

$$287w \equiv 2 \pmod{319} \Rightarrow -10 \times 287w \equiv -10 \times 2 \pmod{319} \quad \text{MI}$$

$$\Rightarrow w \equiv -20 \pmod{319} \quad \text{AI}$$

so

$$w = 299 \quad \text{AI}$$

[5 marks]

Examiners report

Surprisingly, since it is basic bookwork, this part was answered very badly indeed. Most candidates did not realise that -10 was the number to multiply by. Sadly, of the candidates that did do it, some did not read the question carefully enough to see that a positive integer answer was required.

29d. [6 marks]

Markscheme

from (b)

$$-10 \times 861 + 9 \times 957 = 3 \Rightarrow -20 \times 861 + 18 \times 957 = 6 \quad \text{MIAI}$$

so general solution is

$$x = -20 + 319t \quad \text{AIAI}$$

$$y = 18 - 287t \quad (t \in \mathbb{Z}) \quad \text{AIAI}$$

[6 marks]

Examiners report

Again this is standard bookwork. It was answered better than part (c). There were the usual mistakes in the final answer e.g. not having the two numbers, with the parameter, co-prime.

30.

[2 marks]

Markscheme

if p is a prime (and
 $a \equiv 0 \pmod{p}$ with
 $a \in \mathbb{Z}$) then ***AI***
 $a^{p-1} \equiv 1 \pmod{p}$ ***AI***

[2 marks]

Note: Accept
 $a^p \equiv a \pmod{p}$.

Examiners report

Fermat's little theorem was reasonably well known. Some candidates forgot to mention that p was a prime. Not all candidates took the hint to use this in the next part.

31a.

[4 marks]

Markscheme

$$315 = 5 \times 56 + 35 \quad \textbf{MI}$$

$$56 = 1 \times 35 + 21$$

$$35 = 1 \times 21 + 14 \quad \textbf{AI}$$

$$21 = 1 \times 14 + 7$$

$$14 = 2 \times 7 \quad \textbf{AI}$$

$$\text{therefore gcd} = 7 \quad \textbf{AI}$$

[4 marks]

Examiners report

This question was generally well answered although some candidates were unable to proceed from a particular solution of the Diophantine equation to the general solution.

31b.

[9 marks]

Markscheme

(i)

$$7 = 21 - 14 \quad \text{MI}$$

$$= 21 - (35 - 21)$$

$$= 2 \times 21 - 35 \quad \text{(AI)}$$

$$= 2 \times (56 - 35) - 35$$

$$= 2 \times 56 - 3 \times 35 \quad \text{(AI)}$$

$$= 2 \times 56 - 3 \times (315 - 5 \times 56)$$

$$= 17 \times 56 - 3 \times 315 \quad \text{(AI)}$$

therefore

$$56 \times 51 + 315 \times (-9) = 21 \quad \text{MI}$$

$$x = 51, y = -9 \text{ is a solution} \quad \text{(AI)}$$

the general solution is

$$x = 51 + 45N,$$

$$y = -9 - 8N,$$

$$N \in \mathbb{Z} \quad \text{AIAI}$$

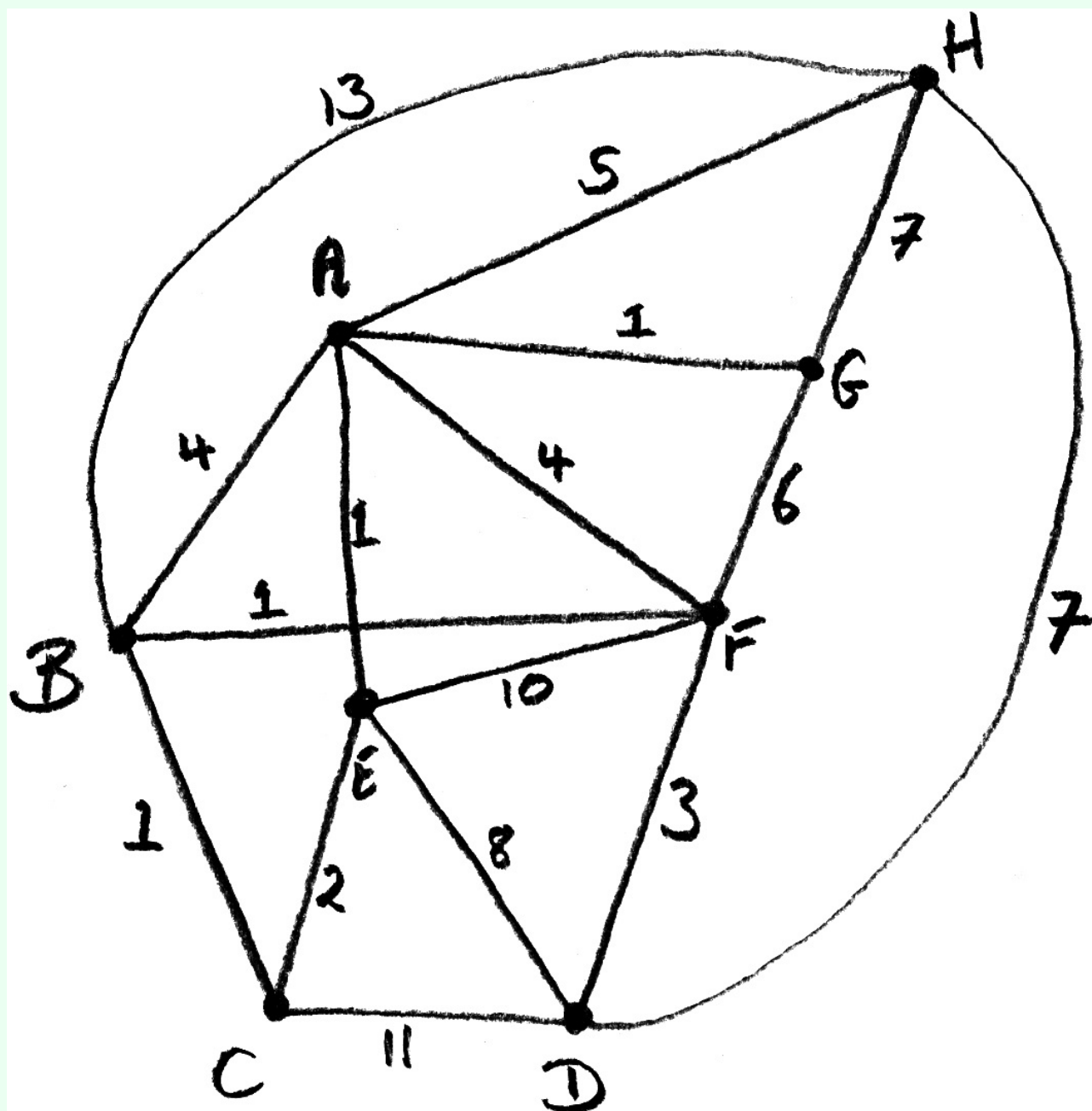
(ii) putting $N = -2$ gives $y = 7$ which is the required value of x **AI**

[9 marks]

Examiners report

This question was generally well answered although some candidates were unable to proceed from a particular solution of the Diophantine equation to the general solution.

Markscheme



A2

Note: Award *A1* if one line missing or one line misplaced. Weights are not required.

[2 marks]

Examiners report

Most candidates were able to draw the graph as required in (a) and most made a meaningful start to applying Prim's algorithm in (b). Candidates were not always clear about the order in which the edges were to be added.

33a.

[4 marks]

Markscheme

$$752 = 2(352) + 48 \quad \text{MI}$$

$$352 = 7(48) + 16 \quad \text{AI}$$

$$48 = 3(16) \quad \text{AI}$$

therefore

$$\gcd(752, 352) \text{ is } 16 \quad \text{RI}$$

[4 marks]

Examiners report

Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates. It was pleasing to see that candidates were not put off by the question being set in context and most candidates were able to start part (b). However, a number made errors on the way, quite a number failed to give the general solution and it was only stronger candidates who were able to give a correct solution to part (b) (iii).

33b.

[10 marks]

Markscheme

(i) let x be the number of cows of breed A

let y be the number of cows of breed B

$$752x + 352y = 8128 \quad \text{AI}$$

(ii)

$16 \mid 8128$ means there is a solution

$$16 = 352 - 7(48) \quad (\text{MI})(\text{AI})$$

$$16 = 352 - 7(752 - 2(352))$$

$$16 = 15(352) - 7(752) \quad (\text{AI})$$

$$8128 = 7620(352) - 3556(752)$$

$$\Rightarrow x_0 = -3556, y_0 = 7620 \quad (\text{AI})$$

$$\Rightarrow x = -3556 + \left(\frac{352}{16}\right)t = -3556 + 22t$$

$$\Rightarrow y = 7620 - \left(\frac{752}{16}\right)t = 7620 - 47t \quad \text{MIAIAI}$$

(iii) for x, y to be

≥ 0 , the only solution is $t = 162 \quad \text{MI}$

$$\Rightarrow x = 8, y = 6 \quad \text{AI}$$

[10 marks]

Examiners report

Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates. It was pleasing to see that candidates were not put off by the question being set in context and most candidates were able to start part (b). However, a number made errors on the way, quite a number failed to give the general solution and it was only stronger candidates who were able to give a correct solution to part (b) (iii).

34a.

[5 marks]

Markscheme

(i) When we sum over the degrees of all vertices, we count each edge twice. Hence every edge adds two to the sum. Hence the sum of the degrees of all the vertices is even. **R2**

(ii) divide the vertices into two sets, those with even degree and those with odd degree **M1**

let S be the sum of the degrees of the first set and let T be the sum of the degrees of the second set

we know $S + T$ must be even

since S is the sum of even numbers, then it is even **R1**

hence T must be even **R1**

hence there must be an even number of vertices of odd degree **AG**

[5 marks]

Examiners report

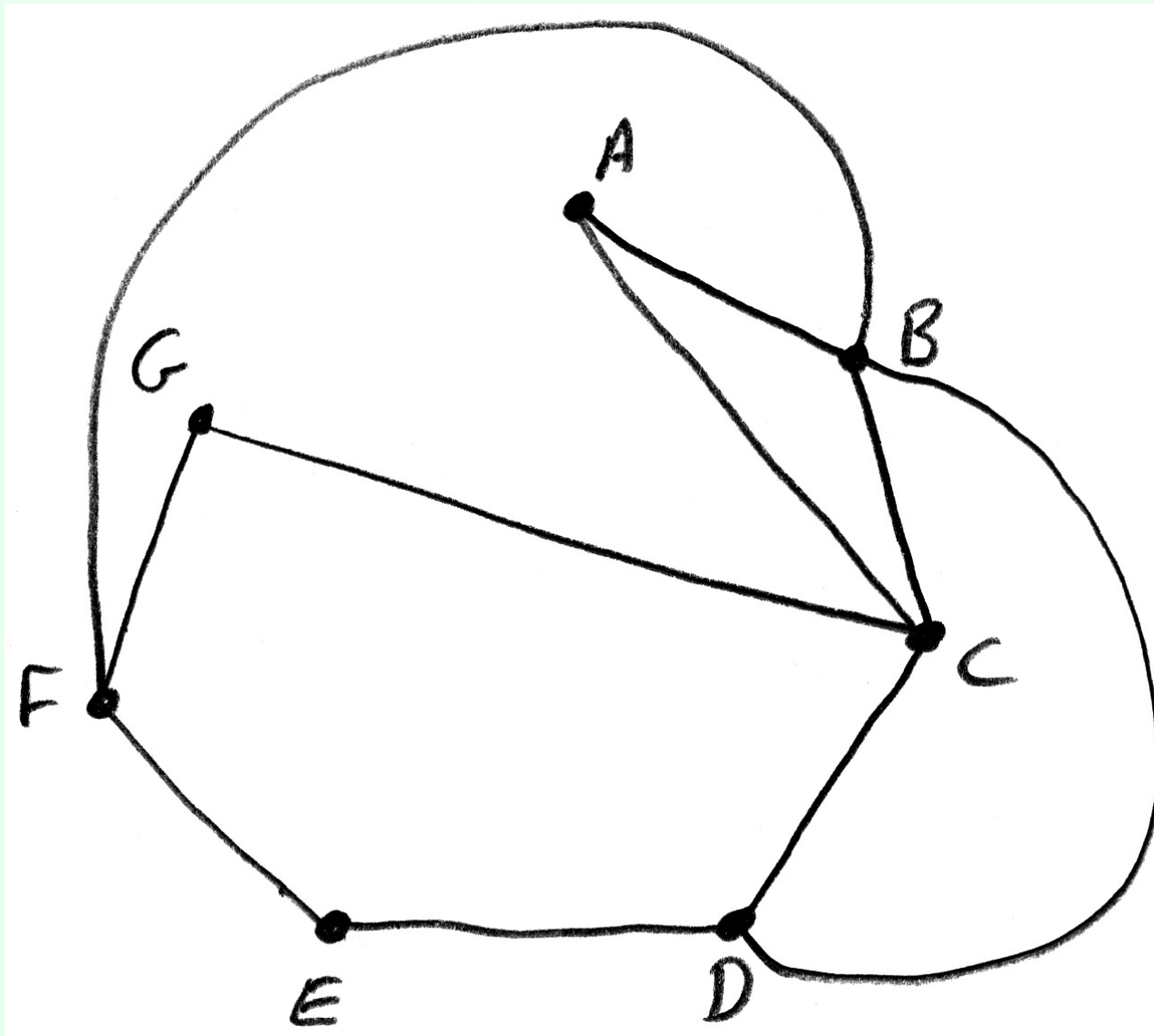
Most candidates were able to start (a), but many found problems in expressing their ideas clearly in words. Stronger candidates had little problem with (b), but a significant number of weaker candidates had problems working with the concepts of Eulerian circuits and Hamiltonian cycles and with understanding how to find a specific number of walks of a certain length as required in (b) (vii).

34b.

[12 marks]

Markscheme

(i)

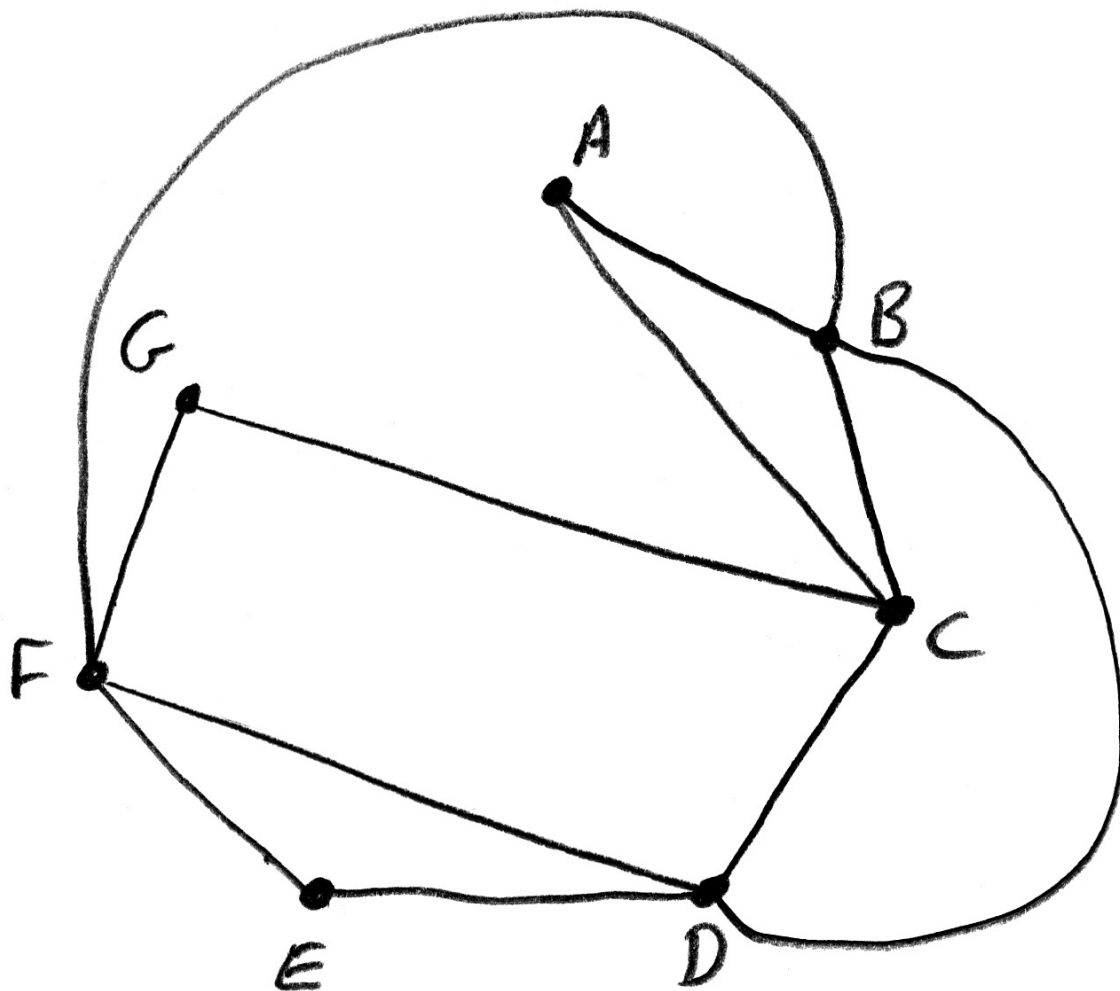


A1

(ii) the graph M is not Eulerian because vertices D and F are of odd degree *A1*

(iii) the edge which must be added is DF *A1*

(iv)



a possible Eulerian circuit is ABDFBCDEFGCA A2

Note: award A1 for a correct Eulerian circuit not starting and finishing at A.

(v) a Hamiltonian cycle is one that contains each vertex in N A1

with the exception of the starting and ending vertices, each vertex must only appear once A1

a possible Hamiltonian cycle is ACGFEDBA A1

(vi)

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad A2$$

(vii) using adjacency matrix to power 4 (M1)

C and F A1

[12 marks]

Examiners report

Most candidates were able to start (a), but many found problems in expressing their ideas clearly in words. Stronger candidates had little problem with (b), but a significant number of weaker candidates had problems working with the concepts of Eulerian circuits and Hamiltonian cycles and with understanding how to find a specific number of walks of a certain length as required in (b) (vii).

35.

[7 marks]

Markscheme

METHOD 1

let x be the number of cars

we know

$$x \equiv 0 \pmod{3} \quad (A1)$$

also

$$x \equiv 4 \pmod{5} \quad (A1)$$

so

$$x = 3t \Rightarrow 3t \equiv 4 \pmod{5} \quad M1$$

$$\Rightarrow 6t \equiv 8 \pmod{5}$$

$$\Rightarrow t \equiv 3 \pmod{5}$$

$$\Rightarrow t = 3 + 5s$$

$$\Rightarrow x = 9 + 15s \quad A1$$

since there must be fewer than 50 cars, $x = 9, 24, 39 \quad A1A1A1$

Note: Only award two of the final three *A1* marks if more than three solutions are given.

[7 marks]

METHOD 2

x is a multiple of 3 that ends in 4 or 9 *R4*

therefore $x = 9, 24, 39 \quad A1A1A1 \quad N3$

Note: Only award two of the final three *A1* marks if more than three solutions are given.

[7 marks]

Examiners report

There were a number of totally correct solutions to this question, but some students were unable to fully justify their results.

36a.

[4 marks]

Markscheme

consider two cases **MI**

let a and p be coprime

$$a^{p-1} \equiv 1 \pmod{p} \quad \mathbf{RI}$$

$$\Rightarrow a^p \equiv a \pmod{p}$$

let a and p not be coprime

$$a \equiv 0 \pmod{p} \quad \mathbf{MI}$$

$$a^p \equiv 0 \pmod{p} \quad \mathbf{RI}$$

$$\Rightarrow a^p \equiv a \pmod{p}$$

so

$$a^p \equiv a \pmod{p} \text{ in both cases} \quad \mathbf{AG}$$

[4 marks]

Examiners report

There were very few fully correct answers. In (b) the majority of candidates assumed that 341 is a prime number and in (c) only a handful of candidates were able to state the converse.

36b.

[7 marks]

Markscheme

$$341 = 11 \times 31 \quad (\mathbf{MI})$$

we know by Fermat's little theorem

$$2^{10} \equiv 1 \pmod{11} \quad \mathbf{MI}$$

$$\Rightarrow 2^{341} \equiv (2^{10})^{34} \times 2 \equiv 1^{34} \times 2 \equiv 2 \pmod{11} \quad \mathbf{AI}$$

also

$$2^{30} \equiv 1 \pmod{31} \quad \mathbf{MI}$$

$$\Rightarrow 2^{341} \equiv (2^{30})^{11} \times 2^{11} \quad \mathbf{AI}$$

$$\equiv 1^{11} \times 2048 \equiv 2 \pmod{31} \quad \mathbf{AI}$$

since 31 and 11 are coprime **RI**

$$2^{341} \equiv 2 \pmod{341} \quad \mathbf{AG}$$

[7 marks]

Examiners report

There were very few fully correct answers. In (b) the majority of candidates assumed that 341 is a prime number and in (c) only a handful of candidates were able to state the converse.

Markscheme

(i) converse: if

$a^p = a \pmod{p}$ then p is a prime ***AI***

(ii) from part (b) we know

$$2^{341} \equiv 2 \pmod{341}$$

however, 341 is composite

hence 341 is a counter-example and the converse is not true ***RI***

[2 marks]

Examiners report

There were very few fully correct answers. In (b) the majority of candidates assumed that 341 is a prime number and in (c) only a handful of candidates were able to state the converse.